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LOCAL ADJUNCTIONS

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The concept of a local adjunction between bicategories $\mathscr{A}$ and $\mathscr{B}$ focuses on a family of adjunctions

$$\mathscr{B}(FA, B) \leftrightarrows \mathscr{A}(A, GB)$$

between hom-categories; hence the name *local* adjunction. When bicategories of spans, or (bi-)modules for some monoidal category are created, ordinary adjunctions induce local adjunctions between the resulting bicategories. Other examples include bi-adjunctions, lax adjunctions, strict quasi-adjunctions and monoidal adjunctions.

Introduction

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between hom-categories; hence the name *local* adjunction. When bicategories of spans [2], or (bi-)modules for some monoidal category [11] are created, ordinary adjunctions induce local adjunctions between the resulting bicategories. Other examples include bi-adjunctions [12], lax adjunctions [4], strict quasi-adjunctions [6] and monoidal adjunctions [9]. Further, given a local adjunction, between distributive bicategories $\mathscr{A}$ and $\mathscr{B}$, which consists of morphisms of bicategories and optransformations, there arises another such between $\mathscr{A}$-mod and $\mathscr{B}$-mod.

This work was motivated by the desire of Kelly to find a more conceptual formulation of some results of Gray's ([7] particularly 2.4.3 and 2.5.3) concerning the effect of a change of base monoidal category on indexed limits. Now, indexed limits can be expressed as representations of homs (also known as liftings and extensions) of (bi-)modules, which in turn are well-behaved under the action of local adjunctions. Later, it became clear that this concept of local adjunction is close to that desired by Betti and Kasangian in [3]. Also, Walters noted in 1984 the relevance of lax adjunctions to the search for a notion of geometric morphism between bicategories. This connection may be pursued in later work.

1. Notation

A module $m: X \nrightarrow Y$ between categories is contravariant in X and covariant in Y; given $n: Y \nrightarrow Z$ the tensor product is written $m \otimes n$, so that the common variable is central. In this style, when a bicategory, such as a base bicategory, is regarded as a generalised bicategory of modules, composition of 1-cells is written algebraically; given $f: A \to B$ and $g: B \to C$, the composite is $f \cdot g: A \to C$. This convention is not followed in every bicategory though, e.g. **Bicat**($\mathcal{A}$, $\mathcal{B}$). Composition of 2-cells, however, is functional, in accord with the usual composition in monoidal categories.

When writing of homs of 1-cells [11] (a generalisation of the left and right ring-module homomorphisms), we use the following convention: with $h: A \to C$, and f and g as above, $A\text{-hom}(f, h): B \to C$ is the universal 1-cell with an (evaluation) 2-cell $\mathrm{ev}: f.A\text{-hom}(f, h) \to h$. The other universal 1-cell is $\text{hom-}C(g, h): A \to B$ with its evaluation $\mathrm{ev}: \text{hom-}C(g, h).g \to h$.

$$\begin{tikzcd} & B \arrow[dr, "A\text{-hom}(f,h)"] \arrow[d, phantom, "\Downarrow"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\ A \arrow[ur, "\text{hom-}C(g,h)"] \arrow[rr, "h"'] & {} & C \end{tikzcd}$$

Let $S: \mathcal{A} \to \mathcal{B}$ be a morphism of bicategories. Recall [5] that a small $\mathcal{A}$-category X consists of a set of objects together with a morphism $e: X \to \mathcal{A}$ where X here also denotes the chaotic bicategory on the set of objects. SX is the $\mathcal{B}$-category having the same objects as X but with morphism $Se: X \to \mathcal{B}$. Likewise, S acts on $\mathcal{A}$-modules and their morphisms (if $m: X \nrightarrow Y$ is an $\mathcal{A}$-module then $Sm: SX \nrightarrow SY$ takes the values $(Sm)(x, y) = Sm(x, y)$).

A *distributive bicategory* [5] is locally cocomplete with colimits stable under horizontal composition. If $\mathcal{A}$ is a distributive bicategory, then there is a bicategory $\mathcal{A}$-mod of small $\mathcal{A}$-categories, $\mathcal{A}$-modules and $\mathcal{A}$-module morphisms. Consequently, we have a morphism of bicategories $S: \mathcal{A}\text{-mod} \to \mathcal{B}\text{-mod}$. Given another morphism T and an optransformation $\eta: S \to T$ there is another $\eta^*: S \to T: \mathcal{A}\text{-mod} \to \mathcal{B}\text{-mod}$ whose components are the modules $\eta^*_X: SX \nrightarrow TX$ which take the values $\eta^*_X(x, x') = \eta_{e(x)} \cdot TX(x, x')$. The 2-cell

$$\begin{tikzcd} SX \arrow[r, "\eta^*_X", "+" marking] \arrow[d] \arrow[dr, phantom, "\overset{\eta^*_m}{\Longrightarrow}"] & TX \arrow[d] \\ SY \arrow[r, "\eta^*_Y"', "+" marking] & TY \end{tikzcd}$$

is the module morphism derived from

$$
\begin{aligned}
Sm(x, y').\eta.TY(y', y) &\xrightarrow{\eta.1} \eta.Tm(x, y').TY(y', y) \\
&\xrightarrow{j.\mu} \eta.TX(x, x).Tm(x, y)
\end{aligned}
$$

where j introduces the identity on x and μ is the right action of Tm. (Note that the unit and associativity 2-cells were not referred to explicitly. These isomorphisms will be suppressed whenever they obscure, rather than clarify, the argument.) In the monoidal case ($\mathscr{A} = \mathscr{V}$, $\mathscr{B} = \mathscr{V}'$), there is a functor $\eta_X : SX \to TX$, from which η_X^* is created. More generally, given a functor $f: X \to Y$, then $f^* = fY: X \nrightarrow Y$ is the module with $fY(x, y) = Y(fx, y)$ and $f_* = Yf: Y \nrightarrow X$ is the module with $Yf(y, x) = Y(y, fx)$. The modules η_{X*} are the components of another optransformation $\eta_* : S^{\mathrm{op}} \to T^{\mathrm{op}} : \mathscr{V}\text{-mod}^{\mathrm{op}} \to \mathscr{V}'\text{-mod}^{\mathrm{op}}$. ($(-)^{\mathrm{op}}$ denotes the reversal of 1-cells, $(-)^{\mathrm{co}}$ denotes the reversal of 2-cells and $(-)^{\mathrm{s}}$ denotes the symmetric reversal.)

Let $F: \mathscr{A} \to \mathscr{B}$ be a morphism of bicategories. Given an object B of $\mathscr{B}$, the double slice category, $F\|B$ is the bicategory which has as objects pairs (A, f) where $f: FA \to B$ is a 1-cell of $\mathscr{B}$; 1-cells are pairs $(a, \chi):(A, f) \to (A', f')$ where $a: A \to A'$ is a 1-cell and $\chi: Fa.f' \to f$ is a 2-cell

$$
\begin{tikzcd}
FA \arrow[d, "Fa"'] \arrow[drr, "f"] & & \\
FA' \arrow[rr, "f'"'] \arrow[r, phantom, shift left=3, "\overset{\chi}{\Longrightarrow}"] & & B
\end{tikzcd}
$$

and 2-cells $\sigma:(a, \chi) \to (a', \chi')$ are 2-cells $\sigma: a \to a'$ of $\mathscr{A}$ such that $(F\sigma.f')\chi' = \chi$.

2. Local adjunctions

Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories, $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{B} \to \mathscr{A}$ be either morphisms or comorphisms and let $\eta = \{\eta_A : A \to GFA\}$ and $\varepsilon = \{\varepsilon_B : FGB \to B\}$ be families of 1-cells indexed by the objects of the bicategories. Often η and ε are the 1-cells of optransformations. Then, for each pair of objects, A in $\mathscr{A}$ and B in $\mathscr{B}$, define the functor

$$\bar{F}_{A,B} : \mathscr{A}(A, GB) \to \mathscr{B}(FA, B)$$

to be $\bar{F}(-) = F(-).\varepsilon$. Similarly, $\bar{G}_{A,B} : \mathscr{B}(FA, B) \to \mathscr{A}(A, GB)$ is $\bar{G}(-) = \eta.G(-)$. Assume now that there are families of natural transformations $\bar{\eta} = \{\bar{\eta}_{A,B}\}$ and $\bar{\varepsilon} = \{\bar{\varepsilon}_{A,B}\}$ making

$$(\bar{F}_{A,B}, \bar{G}_{A,B}, \bar{\eta}_{A,B}, \bar{\varepsilon}_{A,B}) : \mathscr{B}(FA, B) \to \mathscr{A}(A, GB)$$

an adjunction. Then $(F, G, \eta, \varepsilon, \bar{\eta}, \bar{\varepsilon}): \mathscr{B} \to \mathscr{A}$ is a *local adjunction*; alternatively, F is *locally left adjoint* to G. Further, F is the *left adjoint*, η the *unit*, $\bar{F}$ the *local left adjoint*, $\bar{\eta}$ the *local unit* etc. Where no confusion results subscripts will be dropped, and the local adjunction will be denoted $(F, G, \eta, \varepsilon): \mathscr{B} \to \mathscr{A}$.

Example 2.1. A bi-adjunction [12] is a local adjunction (the local adjoints are equivalences).

Example 2.2. Consider monoidal categories as one-object bicategories. Then a monoidal adjunction (the right adjoint is a monoidal functor, the left adjoint comonoidal) is a local adjunction. More generally, if $(F, G, \eta, \varepsilon): \mathscr{B} \to \mathscr{A}$ is a local adjunction such that, for all objects, $GFA = A$ and $FGB = B$ and the components of the unit and counit are identities, then $F_{A,GB} \dashv G_{FA,B}$; $\mathscr{B}(FA, B) \to \mathscr{A}(A, GB)$. Hence $(F^{\mathrm{op}}, G^{\mathrm{op}}, \eta^{\mathrm{op}}, \varepsilon^{\mathrm{op}}, \bar{\eta}, \bar{\varepsilon}): \mathscr{B}^{\mathrm{op}} \to \mathscr{A}^{\mathrm{op}}$ is also a local adjunction.

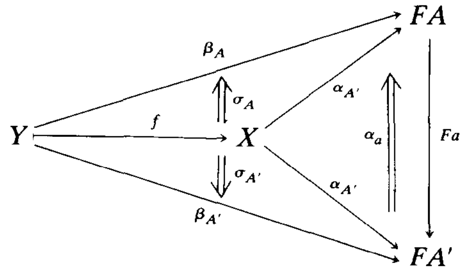
Example 2.3. The strict quasi-adjunctions of Gray [6, I.7.1] are local adjunctions.

Example 2.4. Consider an adjunction $(F, G, \eta, \varepsilon): \mathscr{E} \to \mathscr{F}$ between categories with pullbacks. Then $\mathrm{Sp}(F): \mathrm{Sp}(\mathscr{E}) \to \mathrm{Sp}(\mathscr{F})$ is a comorphism between the categories of spans, $\mathrm{Sp}(G)$ is a homomorphism, and there is a local adjunction $(\mathrm{Sp}(F), \mathrm{Sp}(G), \mathrm{Sp}(\eta), \mathrm{Sp}(\varepsilon)): \mathrm{Sp}(\mathscr{F}) \to \mathrm{Sp}(\mathscr{E})$. Relations are handled similarly.

Example 2.5. Let $(F, G, \eta, \varepsilon, \bar{\eta}, \bar{\varepsilon}): \mathscr{B} \to \mathscr{A}$ be a local adjunction. Then $(G^{\mathrm{s}}, F^{\mathrm{s}}, \varepsilon^{\mathrm{s}}, \eta^{\mathrm{s}}, \bar{\varepsilon}^{\mathrm{op}}, \bar{\eta}^{\mathrm{op}}): \mathscr{A} \to \mathscr{B}$ is a local adjunction.

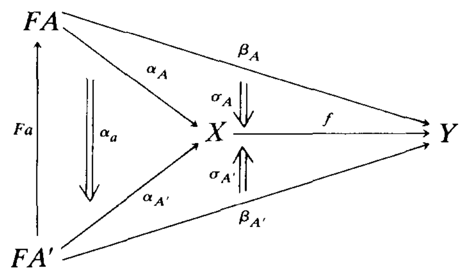
Example 2.6. Let A and C be $\mathscr{V}$-categories where $\mathscr{V}$ is a symmetric monoidal closed category. In [8] (which generalises [3]) are defined 2-functors called behaviour, $B: \mathrm{Cofib}(A, C) \to \mathrm{MOD}(A, C)$ and realization, $R: \mathrm{MOD}(A, C) \to \mathrm{Cofib}(A, C)$. Examination of the proof of Theorem 9 therein shows not only that $\bar{B}^{\mathrm{co}}$ has a right adjoint, but that it is $\bar{R}^{\mathrm{co}}$ for an appropriate choice of η. Hence B^{co} is locally left adjoint to R^{co}.

Example 2.7. Let $F: \mathscr{D} \to \mathscr{A}$ be a morphism of bicategories. Recall that a lax cone α from X to F is an optransformation $\alpha: \Delta X \to F$ where ΔX is the diagonal morphism, and a lax cone morphism from α to β is a modification, $\sigma: \alpha \to \beta$. Then a *limit* for F is a lax cone $\alpha: \Delta X \to F$ such that, for each lax cone $\beta: \Delta Y \to F$ there is a pair (f, σ) where $f: Y \to X$ is a 1-cell and $\sigma: \alpha\Delta f \to \beta$ is a modification, which is universal among such pairs, i.e., given another such pair (f', σ') there is a unique 2-cell $\chi: f' \to f$ such that $\sigma' = \sigma(\alpha\Delta\chi)$.



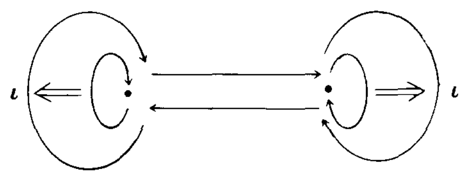
For example, a *local terminal object* in a bicategory $\mathscr{A}$ is a limit for the empty diagram, i.e. an object T with the property that, for each object A of $\mathscr{A}$, there is a terminal object in $\mathscr{A}(A, T)$.

Colimits are defined using the symmetric dual. A typical colimit diagram is



$\Delta: \mathscr{A} \to \mathbf{Bicat}(\mathscr{D}^{\mathrm{op}}, \mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$ is the diagonal morphism of bicategories. $\mathscr{A}$ *has all limits* (respectively *colimits*) *of type* $\mathscr{D}$ if Δ has a local right (resp. left) adjoint.

Example 2.8. Below is an example of a 2-category with two non-equivalent local terminal objects, each of which determines a 2-functor local right adjoint to Δ when $\mathscr{D}$ is empty. Hence, a 2-functor may have two non-equivalent 2-functors as local right adjoints. The 2-graph (omitting identity 2-cells) of this 2-category is given by



Horizontal composition is determined by knowing that non-trivial composites are always the (unique) non-identity 1-cell.

Proposition 2.9. *Let* $(F, G, \eta, \varepsilon): \mathscr{B} \to \mathscr{A}$ *be a local adjunction. If* F *is a morphism, then, for each object* B *of* $\mathscr{B}$, (GB, ε_B) *is a local terminal object in* $F \| B$.

Proof. Given (A, f), an object of $F \| B$, the 1-cell $(\bar{G}f, \bar{\varepsilon}_f):(A, f) \to (GB, \varepsilon_B)$ is terminal since $\bar{F} \dashv \bar{G}$.

$$\begin{tikzcd}
FA \arrow[d, "F\bar{G}f"'] \arrow[drr, "f"] \arrow[drr, phantom, "\overset{\bar{\varepsilon}_f}{\Longrightarrow}" near end, shift right=4ex] & & \\
FGB \arrow[rr, "\varepsilon_B"'] & & B
\end{tikzcd}$$

Of course, there is a dual theorem about initial objects. $\square$

Theorem 2.10. *Let $(F, G, \eta, \varepsilon): \mathscr{B} \to \mathscr{A}$ be a local adjunction where $\mathscr{A}$ and $\mathscr{B}$ are distributive bicategories, F and G are morphisms, and η and ε are optransformations. Then $(F, G, \eta^*, \varepsilon^*): \mathscr{B}\text{-mod} \to \mathscr{A}\text{-mod}$ is a local adjunction.*

Proof. For $m: FX \to Y$, $(\bar{G}m)(x, y) = \bar{G}m(x, y)$. For $n: X \to GY$, $(\bar{F}n)(x, y) \cong \underline{(Fn \otimes \varepsilon^*)(x, y)}$ which is a colimit of terms of the form $\bar{F}n(x, y') \otimes Y(y', y)$. Now $\overline{\eta^*}$ is the morphism of modules given by

$$\begin{aligned}
m(x, y) &\xrightarrow{\bar{\eta}} \bar{G}\bar{F}(m(x, y)) \\
&\xrightarrow{\bar{G}(1 \otimes j)} \bar{G}(\bar{F}m(x, y) \otimes Y(y, y)) \\
&\longrightarrow (\bar{G}\bar{F}m)(x, y)
\end{aligned}$$

(where j introduces the identity). $\overline{\varepsilon^*}: \bar{F}\bar{G}n \to n$ is the module morphism induced by

$$\begin{aligned}
\bar{F}\bar{G}n(x, y') \otimes Y(y', y) &\xrightarrow{\bar{\varepsilon} \otimes 1} n(x, y') \otimes Y(y', y) \\
&\xrightarrow{\mu} n(x, y)
\end{aligned}$$

(where μ is the right action of n). Thus when $\overline{\eta^*}_{\bar{G}}.\bar{G}_{\overline{\varepsilon^*}}$ is expanded into its components, the 1-cell identities introduced in $\overline{\eta^*}$ are cancelled, leaving $\bar{\eta}_{\bar{G}}.\bar{G}_{\bar{\varepsilon}}$ which is an identity 2-cell of $\mathscr{A}$. Similarly, $\bar{F}\overline{\eta^*}.\overline{\varepsilon^*_{\bar{F}}}$ reduces to $\bar{F}\bar{\eta}.\bar{\varepsilon}_{\bar{F}}$. $\square$

Example 2.11. Let $(F, G, \eta, \varepsilon): \mathscr{V} \to \mathscr{V}'$ be a monoidal adjunction where F is a homomorphism. Then the theorem implies $(F, G, \eta^*, \varepsilon^*): \mathscr{V}\text{-mod} \to \mathscr{V}'\text{-mod}$ is a local adjunction. Since $F^{\mathrm{op}} \dashv G^{\mathrm{op}}$ and $\mathscr{V}^{\mathrm{op}}\text{-mod} \cong (\mathscr{V}\text{-mod})^{\mathrm{op}}$ we have another local adjunction $(F^{\mathrm{op}}, G^{\mathrm{op}}, \eta_*, \varepsilon_*): \mathscr{V}\text{-mod}^{\mathrm{op}} \to \mathscr{V}'\text{-mod}^{\mathrm{op}}$. As 1-cells of $\mathscr{V}$-mod, $\overline{G^{\mathrm{op}}}m = Gm \otimes \eta_*$ is denoted by $\underline{G}m$ and $\overline{F^{\mathrm{op}}}$ by $\underline{F}$.

3. Right adjoints, homs and indexed limits

The theorem in this section relates preservation of homs by right adjoints to preservation of composition by left adjoints. A known result on indexed limits then arises in a more natural way.

Theorem 3.1. *Let* $(F, G, \eta, \varepsilon): \mathscr{B} \to \mathscr{A}$ *be a local adjunction and* $f: A \to C$ *be a 1-cell of* $\mathscr{A}$. *If*

(i) *for all* $h: C \to GB$ *we have an isomorphism* $(\tilde{\bar{F}}.1): Ff.\bar{F}h \cong \bar{F}(f.h)$,

then

(ii) *for all* $g: FA \to B$, *if* (FA)-hom(Ff, g) *exists, then* A-hom$(f, \bar{G}g)$ *does and* $\bar{G}(FA)$-hom$(Ff, g) \cong A$-hom$(f, \bar{G}g)$ *naturally in both variables (where applicable).*

$$\begin{tikzcd} & FC \arrow[dr, "(FA)\text{-hom}(Ff,\, g)"] \arrow[d, phantom, "\Downarrow"] & \\ FA \arrow[ur, "Ff"] \arrow[rr, "g"'] & & B \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dr, "A\text{-hom}(f,\, \bar{G}g)"] \arrow[d, phantom, "\Downarrow"] & \\ A \arrow[ur, "f"] \arrow[rr, "\bar{G}g"'] & & GB \end{tikzcd}$$

Hence, if F *is a homomorphism, then* (ii) *holds for all such* f *and* g. *Conversely, if* (FA)-hom(Ff, g) *exists for each* g, *then* (ii) *implies* (i).

Proof. By Yoneda, the following sequence of natural bijections establishes the result:

$$\frac{\dfrac{\dfrac{\dfrac{h \to \bar{G}(FA)\text{-hom}(Ff, g)}{\bar{F}h \to (FA)\text{-hom}(Ff, g)}}{Ff.\bar{F}h \to g}}{\bar{F}(f.h) \to g}}{f.h \to \bar{G}g}$$

For the converse, assume (ii). Then we have another such sequence:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{Ff.\bar{F}h \to g}{\bar{F}h \to (FA)\text{-hom}(Ff, g)}}{h \to \bar{G}(FA)\text{-hom}(Ff, g)}}{h \to A\text{-hom}(f, \bar{G}g)}}{f.h \to \bar{G}g}}{\bar{F}(f.h) \to g} \qquad \square$$

For the rest of this section we consider a monoidal adjunction $(F, G, \eta, \varepsilon): \mathcal{V} \to \mathcal{V}'$. The theorem is clear in this case, but it is worth stating the dual result explicitly for $\mathcal{V}$-mod$^{\mathrm{op}}$.

Corollary 3.2. *Let $m: Z \nrightarrow X$ be a $\mathcal{V}'$-module and $n: Y \nrightarrow FX$ be a $\mathcal{V}$-module. If, for every module $p: GY \nrightarrow Z$, we have an isomorphism $(\tilde{F}.1): \underline{F}p \otimes Fm \to \underline{F}(p \otimes m)$, then $\underline{G}\text{hom-}(FX)(Fm, n) \cong \text{hom-}X(m, \underline{G}n)$ naturally in both variables.*

$$\begin{tikzcd} & FZ \arrow[dr, "Fm"] \arrow[d, phantom, "\Downarrow"] & \\ Y \arrow[ur, "\text{hom-}(FX)(Fm,n)"] \arrow[rr, "n"'] & {} & FX \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow"] & \\ GY \arrow[ur, "\text{hom-}X(m,\, \underline{G}n)"] \arrow[rr, "\underline{G}n"'] & {} & X \end{tikzcd}$$ □

Example 3.3. Consider the monoidal adjunction $(F, U, \eta, \varepsilon): \mathbf{Ab} \to \mathbf{Set}$ where U is the forgetful functor. Let X and Z be one-object **Set**-categories, that is, monoids. A module $m: X \nrightarrow Z$ is then a set with left-X and right-Z actions. FX is the free ring on X and Fm is the free left-F, right-FZ (ring) module. Let Y be a ring and n be a left-FX, right-Y module. Then $\bar{U}(FX)\text{-hom}(Fm, n) \cong X\text{-hom}(m, \bar{U}n)$ i.e., left-FX module homomorphisms from Fm to n are just left-X monoid homomorphisms from m to $\bar{U}n$. A dual statement holds for right module homomorphisms.

Recall that indexed limits are functors which represent homs: for $m: X \nrightarrow Z$ a $\mathcal{V}$-module and $g: X \to Y$ a $\mathcal{V}$-functor $X\text{-hom}(m, gY) \cong \text{colim}(m, g)Y$

$$\begin{tikzcd} & Z \arrow[dr, "X\text{-hom}(m,\, gY)"] \arrow[d, phantom, "\Downarrow"] & \\ X \arrow[ur, "m"] \arrow[rr, "gy"'] & {} & Y \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[dr, "\text{colim}(m,\, g)"] & \\ X \arrow[ur, "m"] \arrow[rr, "g"'] & & Y \end{tikzcd}$$

and for $h: Z \to Y$ another $\mathcal{V}$-functor, $\text{hom-}Z(m, Yh) \cong Y\text{lim}(m, h)$

$$\begin{tikzcd} & X \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow"] & \\ Y \arrow[ur, "\text{hom-}Z(m,Yh)"] \arrow[rr, "Yh"'] & {} & Z \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dr, "m"] \arrow[dl, "\text{lim}(m,h)"'] & \\ Y & & Z \arrow[ll, "h"] \end{tikzcd}$$

In order to interpret the theorem for indexed limits, we need first to record some notation and elementary facts. Assume for the rest of the section that $(F, G, \eta, \varepsilon): \mathcal{V} \to \mathcal{V}'$ is a monoidal adjunction. For $g: FX \to Y$ a $\mathcal{V}$-functor write $\bar{G}g$ for $\eta.Gg$.

Lemma 3.4. *Let $m: X \nrightarrow Z$ be a $\mathcal{V}'$-module, and $g: FX \to Y$ and $h: X' \to Y$ be $\mathcal{V}$-functors. Then*

$$G(hY) = (Gh)GY\,, \tag{3.1}$$

$$G(Yh) = GY(Gh)\,, \tag{3.2}$$

$$\bar{G}(gY) \cong (\bar{G}g)GY\,, \tag{3.3}$$

$$\underline{G}(Yg) \cong GY(\bar{G}g)\,, \tag{3.4}$$

$$(Fm)^{\mathrm{op}} = F^{\mathrm{op}}m^{\mathrm{op}}\,. \quad \square \tag{3.5}$$

Corollary 3.5. *Let F be a homomorphism and $f: FZ \to Y$ be a $\mathcal{V}$-functor, with g and m as above. Then, if* $\operatorname{colim}(Fm, g)$ *exists, then* $\operatorname{colim}(m, \bar{G}g)$ *does and* $\bar{G}\operatorname{colim}(Fm, g) \cong \operatorname{colim}(m, \bar{G}g)$ *naturally in both variables. Dually, if* $\lim(Fm, f)$ *exists, then* $\lim(m, \bar{G}f)$ *does and* $\underline{G}\lim(Fm, f) \cong \lim(m, \bar{G}f)$ *naturally in both variables.*

Proof. Apply the lemma, definition and theorem in $\mathcal{V}'$-mod to show $[\bar{G}\operatorname{colim}(Fm, g)]GY$ is isomorphic to $[X\text{-hom}m, (\bar{G}g)]GY$. The dual follows similarly. $\square$

The simplest type of indexed limit occurs when X or Z is $\mathcal{I}$, (the $\mathcal{V}'$-category with one object whose hom is the unit of $\mathcal{V}'$). Then a module $m: X \twoheadrightarrow \mathcal{I}$ (respectively, a module $m: \mathcal{I} \twoheadrightarrow Z$) is equivalent, if $\mathcal{V}'$ is a $\mathcal{V}'$-category, to a functor $f: X^{\mathrm{op}} \to \mathcal{V}'(f: Z \to \mathcal{V}')$. For $g: X \to Y$ ($g: Z \to Y$) the colimit (limit) of g indexed by f is now an object of Y denoted $f * g$ ($\{f, g\}$).

Conceptual difficulties for limits indexed by functors then arise in considering a change of base since, unlike the module case, the image under F of a $\mathcal{V}$-valued functor is not a $\mathcal{V}'$-valued functor; for $f: Z \to \mathcal{V}'$ we have $Ff: FZ \to F\mathcal{V}'$. When F is a closed functor, the ad-hoc remedy is to compose Ff with $\hat{F}: F\mathcal{V} \to \mathcal{V}'$ (the monoidal functor whose action on hom objects is $\hat{F}: F[V, W] \to [FV, FW]$).

Corollary 3.6 (Gray). *Let $(F, G, \eta, \varepsilon): \mathcal{V} \to \mathcal{V}'$ be a monoidal closed adjunction and $g: FX \to Y$ be a $\mathcal{V}$-functor. Then, if $f: X^{\mathrm{op}} \to \mathcal{V}'$ is a $\mathcal{V}'$-functor such that $(Ff.\hat{F}) * g$ exists, then $f * \bar{G}g$ exists and $(Ff.\hat{F}) * g$, as an object in GY, is isomorphic to $f * \bar{G}g$. Dually, if $f: X \to \mathcal{V}'$ is a $\mathcal{V}'$-functor such that $\{Ff.\hat{F}, g\}$ exists, then $\{f, \bar{G}g\}$ exists and $\{Ff.\hat{F}, g\}$, as an object in GY, is isomorphic to $\{f, \bar{G}g\}$.* $\square$

4. Lax adjunctions

Lax left adjoints to 2-functors were defined in [4] in terms of an uninterpreted list of equations. They were used to describe the algebras (e.g. relational algebras in [1]) for lax monads. Here a lax adjunction is a local adjunction whose local units and counits are related by horizontal composition. Theorem 4.1 demonstrates that when a comorphism is local left adjoint to a 2-functor, the two concepts agree. In this way we can construct, not only monads built from comorphisms, but from morphisms: comonads of both kinds may also be built. A *lax adjunction* $(F, G, \eta, \varepsilon, \bar{\eta}, \bar{\varepsilon}): \mathcal{B} \to \mathcal{A}$ is a local adjunction for which the unit and counit are optransformations, and which satisfies one of the following pairs of conditions:

(i) F and G are comorphisms,

$$\begin{aligned} f.h &\xrightarrow{\bar{\eta}} \bar{G}\bar{F}(f.h) \\ &\xrightarrow{\bar{G}(\tilde{F}.1)} \bar{G}(Ff.\bar{F}h) \\ &\xrightarrow{1.\tilde{G}} \bar{G}Ff.G\bar{F}h \end{aligned} \tag{4.1}$$

is

$$\begin{aligned} f.h &\xrightarrow{1.\bar{\eta}} f.\bar{G}\bar{F}h \\ &\xrightarrow{\eta_f.1} \bar{G}Ff.G\bar{F}h \end{aligned}$$

and

$$\bar{F}\bar{G}(k.g) \xrightarrow{\bar{\varepsilon}} k.g$$

is

$$\begin{aligned} \bar{F}\bar{G}(k.g) &\xrightarrow{\bar{F}(1.\tilde{G})} \bar{F}(\bar{G}k.Gg) \\ &\xrightarrow{\tilde{F}.1} F\bar{G}k.\bar{F}Gg \\ &\xrightarrow{1.\varepsilon_g} F\bar{G}k.\varepsilon g \\ &\xrightarrow{\bar{\varepsilon}.1} k.g\,; \end{aligned} \tag{4.2}$$

or, (ii) F and G are morphisms,

$$\bar{\eta} = \bar{G}(\tilde{F}.1)(1.\tilde{G})(\eta_f.1)(1.\bar{\eta}): f.h \to \bar{G}\bar{F}(f.h) \tag{4.3}$$

and

$$\bar{\varepsilon}\bar{F}(1.\tilde{G})(\tilde{F}.1) = (\bar{\varepsilon}.1)(1.\varepsilon_g): F\bar{G}k.\bar{F}Gg \to k.g\,. \tag{4.4}$$

Note that the two types of lax adjunction are dual, i.e. $(F^{\mathrm{s}}, G^{\mathrm{s}}, \eta^{\mathrm{s}}, \varepsilon^{\mathrm{s}}): \mathcal{A} \to \mathcal{B}$ is also a lax adjunction of the opposite type. When F is a comorphism and G a morphism, then conditions like (4.1) and (4.2) could be stated to define a lax adjunction. Against such generality are ranged the following arguments: in this case η and ε are not optransformations; there is no corresponding theory of monads; and, no equations can be given when F is a morphism and G a comorphism.

Clearly, monoidal adjunctions are lax and an easy computation shows that Sp(–) of an adjunction is lax, too. It is not yet clear to the author whether bi-adjunctions, or even bi-equivalences need be lax adjunctions. When lax adjunctions are lifted to bicategories of modules they remain lax.

If a comorphism F is a lax left adjoint to a homomorphism G, as is the case with spans or monads, then η and ε are determined by their actions on identities: by defining $n = \bar{G}(F^0.1)\bar{\eta}: \iota \to \bar{G}\varepsilon$, and $e = \bar{\varepsilon}\bar{F}(1.G^0)\rho: \bar{F}\eta \to \iota$, and employing

(4.1) and (4.2), $\bar{\eta}$ is seen to be

$$
\begin{aligned}
f &\xrightarrow{\rho} f.\iota \\
&\xrightarrow{1.n} f.\eta_G.G\varepsilon \\
&\xrightarrow{n_f.1} \eta.GFf.G\varepsilon \\
&\xrightarrow{1.\tilde{G}} \bar{G}\bar{F}f
\end{aligned}
\tag{4.5}
$$

and $\bar{\varepsilon}$ is

$$
\begin{aligned}
\bar{F}\bar{G}g &\xrightarrow{\tilde{F}} F\eta.FGg.\varepsilon \\
&\xrightarrow{\varepsilon_g} F\eta.\varepsilon_F.g \\
&\xrightarrow{e.1} \iota.g \\
&\xrightarrow{\lambda^{-1}} g\,.
\end{aligned}
\tag{4.6}
$$

(λ and ρ are the isomorphisms introducing the identities.) The adjunction equations on $\bar{\eta}$ and $\bar{\varepsilon}$ may now be dismantled in terms of n and e.

Theorem 4.1. *Let $F:\mathcal{A}\to\mathcal{B}$ be a comorphism, $G:\mathcal{B}\to\mathcal{A}$ a homomorphism and $\eta:1\to GF$ and $\varepsilon:FG\to 1$ be optransformations. Then $(F, G, \eta, \varepsilon):\mathcal{B}\to\mathcal{A}$ is a lax adjunction iff there is a pair of families of 2-cells $n:\iota\to \bar{G}\varepsilon$ and $e:\bar{F}\eta\to\iota$ such that, with $\bar{\eta}$ and $\bar{\varepsilon}$ defined as above, the following equations hold:*

$$\bar{G}e\bar{\eta} = (1.G^0)\rho : \eta \to \bar{G}i\ , \tag{4.7}$$

$$\bar{G}\varepsilon_g\bar{\eta} = (1.\tilde{G})(n.1)\lambda : Gg \to \bar{G}(\varepsilon.g)\ , \tag{4.8}$$

$$\bar{\varepsilon}_\varepsilon\bar{F}n = \lambda^{-1}(F^0.1) : \bar{F}\iota \to \varepsilon\ , \tag{4.9}$$

$$\bar{\varepsilon}_{Ff}\bar{F}\eta_f = (1.e)(\tilde{F}.1) : \bar{F}(f.\eta) \to Ff\ . \tag{4.10}$$

Proof. Assume that the equations hold. Then (4.1) and (4.2) follow directly and so it is sufficient to prove the adjunction equations for $\bar{\eta}$ and $\bar{\varepsilon}$, which, by symmetry, reduces to proving $\bar{G}\bar{\varepsilon}.\bar{\eta}_{\bar{G}} = 1$. Now apply (4.1) to $\bar{\eta}$ followed by (4.8) and (4.7). Conversely, assume F is locally left adjoint to G. Define n and e as above. Then (4.7) holds since $\bar{G}\bar{\varepsilon}_\iota\bar{\eta}_{\bar{G}\iota} = 1$. Also, (4.8) follows since the transpose under the local adjunction of the left-hand side is ε_g, while that of the right-hand side is $\varepsilon_g\bar{\varepsilon}_{\bar{F}}\bar{F}\bar{\eta}$. The other equations are dual. $\square$

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